# DETERMINATION OF MEAN ELEMENTS FOR VINTI'S SATELLITE THEORY

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Abstract. The method developed by Vinti for predicting the motion of an artificial satellite is used to obtain the first order partial derivatives in the conditional equations. These in turn are used in orbit improvement programs to determine the mean elements of Vinti's theory, corresponding to a separable Hamiltonian. The results have been programmed and tested on the IBM 7094 Mod. II.

## INTRODUCTION

The kinetic equations of motion (Vinti 1959) constitute a reduction of the problem of artificial satellite motion to quadratures. Solution for the reference orbit consists of inverting these equations to find the coordinates $\rho$ and $\eta$ as functions of time. The $\alpha_i$ and $\beta_i$ (i = 1, 2, 3) are the Jacobi constants. If the initial conditions are known, one can readily evaluate the $\alpha_i$. Then if one can evaluate the integrals, one can evaluate the $\beta_i$. Evaluating the integrals and consequently performing the required inversion depend on factoring the quartics $F(\rho)$ and $G(\eta)$. These are given by

$$
\begin{aligned}
F(\rho) &= c^2 \alpha_3^2 + (\rho^2 + c^2)(-\alpha_2^2 + 2\mu\rho + 2\alpha_1\rho^2) \\
&= (-2\alpha_1)(\rho - \rho_1)(\rho_2 - \rho)(\rho^2 + A\rho + B)
\end{aligned}
$$

$$
\begin{aligned}
G(\eta) &= -\alpha_3^2 + (1 - \eta^2)(\alpha_2^2 + 2\alpha_1 c^2 \eta^2) \\
&= -2\alpha_1 c^2 (\eta_0^2 - \eta^2)(\eta_2^2 - \eta^2)
\end{aligned}
$$

Comparison of the two expressions for $F(\rho)$ leads to four algebraic equations for the quantities $\rho_1$, $\rho_2$, A, and B in terms of $\alpha_1$, $\alpha_2$, $\alpha_3$, $\mu$, and $c^2$. Here we use the constants of the motions $\beta_1$, $\beta_2$, $\beta_3$, and

$$a_0 = \frac{-\mu}{2\alpha_1},$$

$$e_0 = \left[1 + \frac{2\alpha_1\alpha_2^2}{\mu^2}\right],$$

$$i_0 = \cos^{-1}\left(\frac{\alpha_3}{\alpha_2}\right),$$

obtainable from initial conditions. One can then find $\eta_0$ and $\eta_2$ explicitly, and $\rho_1$, $\rho_2$, A, and B, by successive approximations through order $J_2^2$, where $J_2$ is the coefficient of the second zonal harmonic of the planet's gravitational potential.

There is another set of elements introduced by Izsak (Izsak 1960) which result in the immediate factoring of the quartics $F(\rho)$ and $G(\eta)$. These are $\beta_1$, $\beta_2$, $\beta_3$, and

$$a = \frac{1}{2}(\rho_1 + \rho_2),$$

$$e = \frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)},$$

$$I = \sin^{-1}\eta_0$$

It is possible to express $\alpha_1$, $\alpha_2$, $\alpha_3$, $\eta_0$, $\eta_2$, $\rho_1$, $\rho_2$, A, and B explicitly in terms of a, e, and I. To obtain a, e, and I, we first approximate them by $a_0$, $e_0$, and $i_0$, compute the orbit by the above method, compare results with observations over an interval of time, find residuals, and then do iterated differential corrections (Bonavito 1962, 1964).

## DIFFERENTIAL COEFFICIENTS

Since the integrands contain double valued functions, Vinti has introduced a set of uniformizing variables E, v, and $\psi$, to simplify them. These are defined as variables analogous to the eccentric anomaly, true anomaly, and argument of latitude respectively.

Let $L_0$ denote any observed quantity characteristic of the method of tracking of the satellite. For example, this can represent a direction cosine for the minitrack radio system, a range rate for doppler tracking, or a right ascension or declination for optical tracking. Denoting a, e, $\eta_0$, $\beta_1$, $\beta_2$, and $\beta_3$ by $q_i$ (i = 1, 2, . . . 6), we can then write the conditional equations as

$$\Delta L = L_0 - L_c = \sum_{i=1}^{6} \left(\frac{\partial L_c}{\partial q_i}\right) \Delta q_i$$

Here $L_c$ represents the computed value of the observed quantity. The first derivatives are given by the equation

$$\frac{\partial L_c}{\partial q_i} = \frac{\partial L_c}{\partial x_m}\frac{\partial x_m}{\partial q_i} + \frac{\partial L_c}{\partial y_m}\frac{\partial y_m}{\partial q_i} + \frac{\partial L_c}{\partial z_m}\frac{\partial z_m}{\partial q_i}$$

With use of the appropriate transformations, the expressions obtained for $L_c$, $\partial L_c/\partial x_m$, $\partial L_c/\partial y_m$, and $\partial L_c/\partial z_m$, will depend only on the local coordinates. One can then determine them from the values of the inertial coordinates x, y, z, computed by the Vinti orbit generator for specific observation times.

By means of the equations

$$\rho = a(1 - e \cos E),$$

$$\eta = \eta_0 \sin \psi,$$

the inertial coordinates can be written

$$x = \sqrt{[a^2 (1 - e \cos E)^2 + c^2] (1 - \eta_0^2 \sin^2 \psi)} \cos \phi,$$

$$y = \sqrt{[a^2 (1 - e \cos E)^2 + c^2] (1 - \eta_0^2 \sin^2 \psi)} \sin \phi,$$

and

$$z = a(1 - e \cos E) \eta \sin \psi.$$

One can derive the formulas for $\partial x_m/\partial q_i$, $\partial y_m/\partial q_i$, and $\partial z_m/\partial q_i$ in terms of $\rho$, $\eta$, and $\phi$ by direct differentiation of x, y, z, with respect to a, e, $\eta_0$, $\beta_1$, $\beta_2$, and $\beta_3$. Since the coordinates x, y, and z are functions of the uniformizing variables, it is necessary to obtain the expressions for $\partial E/\partial q_i$, and $\partial \psi/\partial q_i$ (Bonavito 1964).

This can be performed in the following manner:

Differentiating the expressions on page 199 of reference 5 (Vinti 1961), we list

1. $\dfrac{\partial M_s}{\partial q_i}$ and $\dfrac{\partial \psi_s}{\partial q_i}$

2. $\dfrac{\partial E_0}{\partial q_i}$ from the Kepler equation.

3. $\frac{\partial v_0}{\partial q_i}$ by using $v = M_s + v_0$, $E = M_s + E_0$, and the anomaly connections.

4. $\frac{\partial \psi_0}{\partial q_i}$, $\frac{\partial M_1}{\partial q_i}$, and $\frac{\partial E_1}{\partial q_i}$

5. $\frac{\partial v_1}{\partial q_i}$ by using $v = M_s + v_0 + v_1$, $E = M_s + E_0 + E_1$, and the anomaly connections.

6. $\frac{\partial \psi_1}{\partial q_i}$, $\frac{\partial M_2}{\partial q_i}$, and $\frac{\partial E_2}{\partial q_i}$, which gives $\frac{\partial E}{\partial q_i} = \frac{\partial(M_s + E_0 + E_1 + E_2)}{\partial q_i}$

7. Then $\frac{\partial v_2}{\partial q_i}$ with use of $v = M_s + v_0 + v_1 + v_2$, $E = M_s + E_0 + E_1 + E_2$, and the anomaly connections, which gives

$$\frac{\partial v}{\partial q_i} = \frac{\partial(M_s + v_0 + v_1 + v_2)}{\partial q_i}$$

8. Then $\frac{\partial \psi_2}{\partial q_i}$ yields $\frac{\partial \psi}{\partial q_i} = \frac{\partial(\psi_s + \psi_0 + \psi_1 + \psi_2)}{\partial q_i}$

9. $\frac{\partial \chi}{\partial q_i}$, where $\chi$ is an angle that equals $\psi$ whenever $\psi$ is a multiple of $\frac{\pi}{2}$, and which also satisfies the equation

$$\tan \chi = (1 - \eta_0^2)^{1/2} \tan \psi$$

10. Finally $\frac{\partial \phi}{\partial q_i}$

Since the quantities $\partial E/\partial q_i$, $\partial\psi/\partial q_i$, and $\partial\phi/\partial q_i$ are obtained only from the theoretical calculation, one can then immediately determine the first derivatives for use with any method of tracking whatsoever after deciding on the form of $L_c$ and its partial derivative with respect to the local coordinates (Bonavito 1964). Corrections to the orbital elements $\Delta q_i$ are computed by an iterated least square procedure after a sufficient number of observations are obtained, a minimum of six conditional equations being necessary to determine the problem. Classically speaking, the initial coordinates of the system point are shifted in p, q phase space such that the continuous evolution of a canonical transformation with this new value of the Hamiltonian produces a motion of a mechanical system consistent with observation, in this case the satellite orbit.

In the following, $\sigma = \sqrt{\Sigma(D_0 - D_c)^2/(N_1 - N_2)}$ where $D_0$ and $D_c$ are the observed and computed comparison parameters (i.e.: direction cosines, right ascension, declination, etc.), $N_1$ is the number of equations of condition (20), and $N_2$ is the number of variables (6). Figure 1 is a plot over an allowed ten iterations of the standard deviation of fit $\sigma$, times $10^3$ for direction cosine observation data from the Relay II Satellite (#64031). The criterion for accepting observations is $\Delta L$ $(\Delta M) \leq 3 \times 10^{-3}$. The twenty equations of condition (without weighting factors) used in a least squares routine to find the six Izsak elements, extend over an arc of three hours. Figure 2 is a similar plot for Smithsonian Astrophysical Observatory optical data from the ANNA Satellite. Twenty equations of condition are used for an arc of seventy five hours following injection, and the criterion for accepting observations is $\Delta L$ $(\Delta M) \leq 20 \times 10^{-3}$ milliradians. Although the number of iterations was extended to ten, the program converged after the third iteration for the Relay II data, and after the second iteration for the data from Satellite ANNA. The value of the eccentricity for each of the above satellites is considerably different. Thus, for the initial epoch, they were taken to be 0.23597617 for Relay II, and 0.00671710 for ANNA. Figure 3 is a plot for the first seven equations of condition of Satellite ANNA covering a period of the first forty four hours following injection. While a convergence criterion of $0.2 \times 10^{-3}$ for the standard deviation of fit was met using the ANNA data and twenty equations of condition, the theory was able to meet a value of $0.04 \times 10^{-3}$ for the case of seven equations of condition. Also, it has been shown that the Vinti orbit generating program computes positions and velocities very rapidly and with great accuracy (Bonavito 1962). For seventy equations of condition of the Relay II Satellite extending over a five hour period, the program converged on the third iteration to a standard deviation of fit criterion of $2.7 \times 10^{-3}$ within thirty seconds. All of the above investigations were conducted on the IBM 7094 electronic digital computer. These results are based only on the Vinti reference potential, without the effects of the third harmonic or the residual fourth harmonic. Inclusion of these other effects, which could be accomplished by use of the perturbation theory of Vinti (1963) would be expected to furnish computed orbits which would agree with observation over a longer interval of time.

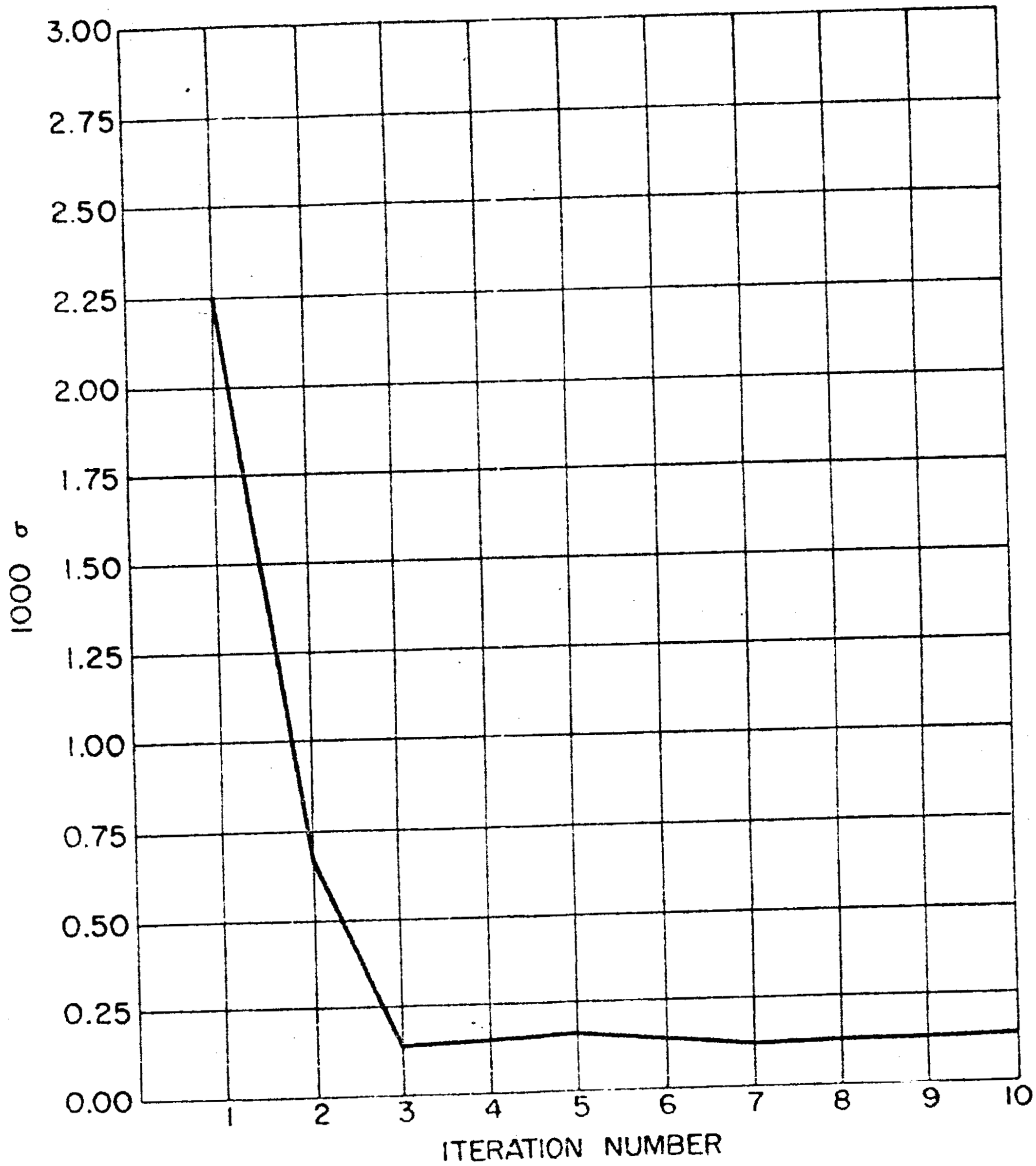
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
1000 σ
1
2
3
4
5
6
7
8
9
10
ITERATION NUMBER

FIGURE 1

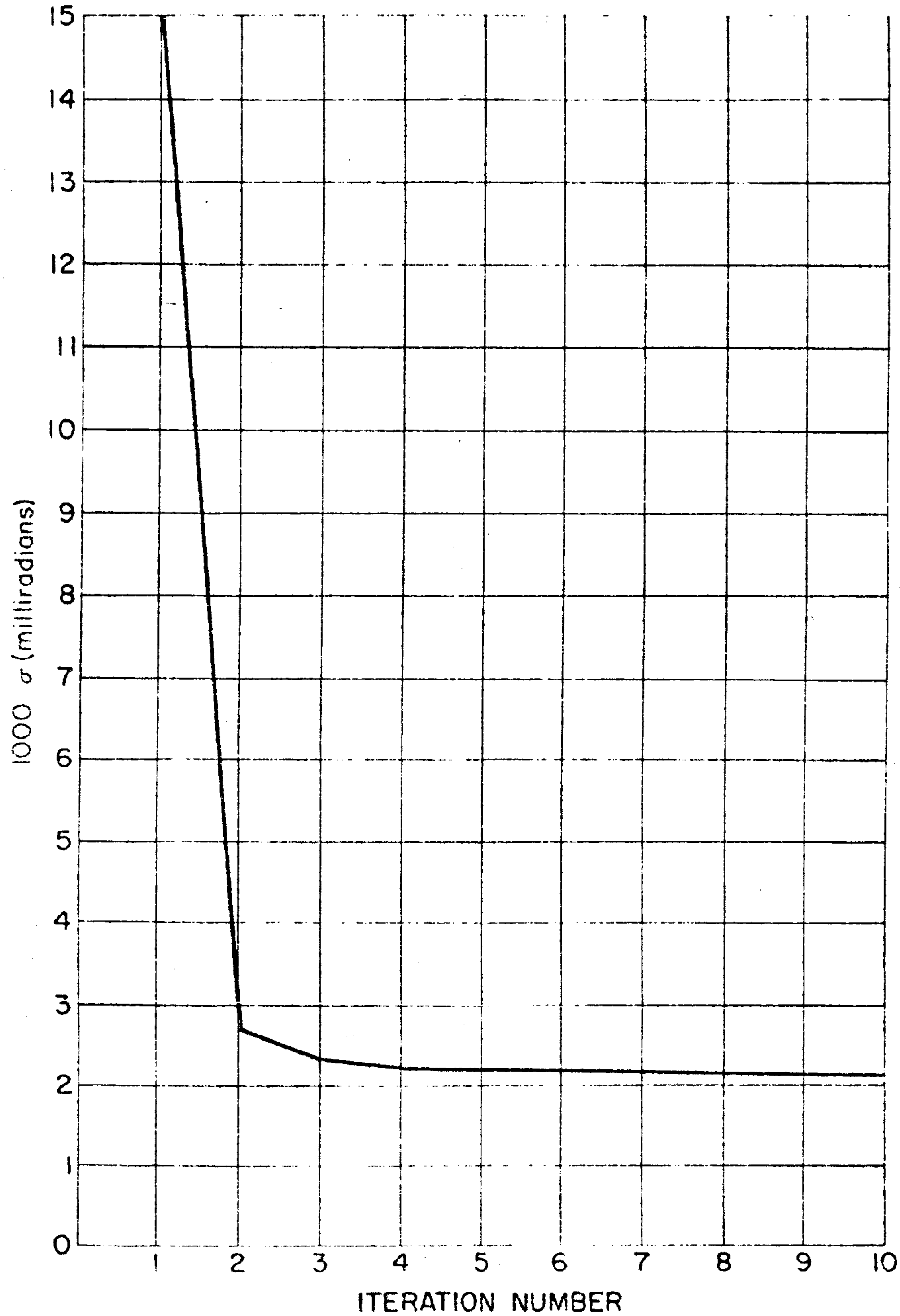
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
1000 σ (milliradians)
1
2
3
4
5
6
7
8
9
10
ITERATION NUMBER

FIGURE 2

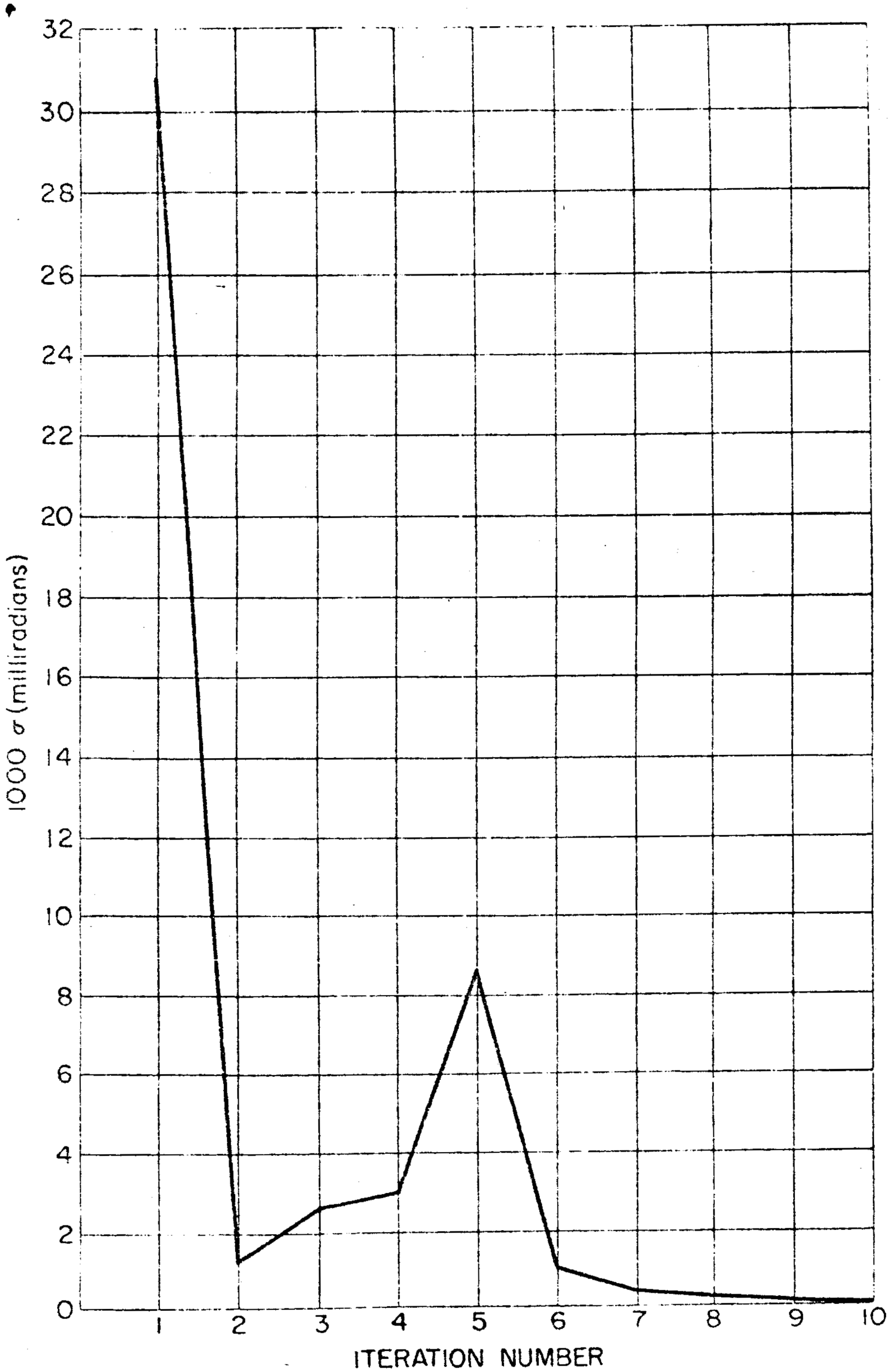
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
1000 σ (milliradians)
1
2
3
4
5
6
7
8
9
10
ITERATION NUMBER

FIGURE 3

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